

Lecture 14

- kramers - kronig relations, dispersion
- Propagation of a charged particle through media.

• We return to the study of EM fields in media, but at a more advanced level

→ absorption

→ non-locality in time

→ convolution and frequency

→ analytic properties

→ Kramers-Kronig relations

→ example: resonant absorption model. →

→ very general!

• Let us remember the following effect:
polarisation of dielectric media
by external electric field

$$\vec{P}(\vec{x}) = \sum_n \vec{d}_n \cdot f(\vec{x}_n - \vec{x}) \quad \leftarrow \begin{array}{l} \text{smearing} \\ \text{function} \end{array}$$

↓ electric polarization of the medium.

$$\langle \eta \rangle(\vec{x}) = \rho(\vec{x}) - \vec{\nabla}_x \cdot \vec{P}(\vec{x})$$

From this we derived averaged Maxwell equations:

$$\vec{J} \cdot \vec{E} = \frac{\rho}{\epsilon_0} - \frac{\vec{\nabla} \cdot \vec{P}}{\epsilon_0}$$

We also defined the Displacement vector

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P},$$

And we did a similar treatment for the magnetic phenomenon

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \quad \leftarrow \text{magnetisation}$$

The Maxwell equations eventually read:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

To make use of them we need the relation between \vec{E} and \vec{D} and \vec{H} and \vec{B} which we chose as

$$\vec{D}(\vec{x}, t) = \epsilon \vec{E}(\vec{x}, t)$$

$$\vec{H}(\vec{x}, t) = \frac{1}{\mu} \vec{B}(\vec{x}, t)$$

This result is correct for static fields, however, we also used it for our discussion of EM waves. In the case of time-dependant fields there are two new effects:

→ absorption (ϵ and μ have imaginary parts)

→ dispersion (relation between \vec{E} and \vec{D} (\vec{H} and \vec{B}) is non-local in time)

The two effects turn out to be closely related.

• Let us first understand why imaginary parts of ϵ and μ lead to absorption of energy of EM fields by the medium.

Let us consider a monochromatic plane wave in a material:

$$\vec{E} = \frac{1}{E} e^{i\omega t + i\vec{k} \cdot \vec{x}}$$

suppose $\vec{k} = (k_x, 0, 0)$

$$\text{then } k_x = \sqrt{\epsilon_1} \omega \frac{1}{c}$$

If ϵ or μ have imaginary part then k_x has imaginary part \Rightarrow wave exponentially decays

From now on we will focus on ϵ , assuming μ real and constant. Then a general relation between D and E is

$$D(t) = \epsilon_0 E(t) + \epsilon_0 \int_0^{\infty} \chi(\tau) E(t-\tau) d\tau$$

Here we singled out the first term

for convenience, but most importantly, integrated is over positive $\tau \Rightarrow$ causality!

Let us now Fourier transform this expression:

$$D(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D(t) e^{-i\omega t} dt$$

$$D(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} D(t) e^{i\omega t} dt$$

$$D(\omega) = \epsilon_0 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega t} E(t) dt +$$

$$+ \epsilon_0 \int_0^{\infty} d\tau \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{i\omega t} f(\tau) E(t-\tau) =$$

$$= \epsilon_0 E(\omega) + \epsilon_0 E(\omega) \int_0^{\infty} d\tau f(\tau) e^{i\omega\tau} \equiv$$

$$\equiv \epsilon(\omega) E(\omega)$$

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \int_0^{\infty} d\tau f(\tau) e^{i\omega\tau} \quad (*)$$

$\epsilon(\omega)$ has real and imaginary parts:

$$\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$$

Because $f(\tau)$ is real

$$\epsilon(-\omega) = \epsilon^*(\omega):$$

$$\epsilon'(-\omega) = \epsilon'(\omega) \text{ and } \epsilon''(-\omega) = -\epsilon''(\omega)$$

In dielectrics when $\omega \rightarrow 0$ we get the static situation (fields do not depend on time) ϵ becomes equal to real static susceptibility.

Causality and analytic properties of $\epsilon(\omega)$

• We will now consider ϵ as a function of complex variable ω , like we did for the Green's function of \square operator.

• Because integral in (*) starts at $\tau=0$ $\epsilon(\omega)$ is an analytic function in the upper half plane: integral converges if $\text{Im } \omega > 0$ $e^{i\omega\tau} \sim e^{-(\text{Im } \omega)\tau}$.

We assume that the "memory" is finite so integral also converges for real ω (it is not true for conductors for $\omega=0$)

For very large real ω we can show that
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\text{const}}{\omega^2} + \dots$$

This is because at very high frequencies charged particles are basically free.

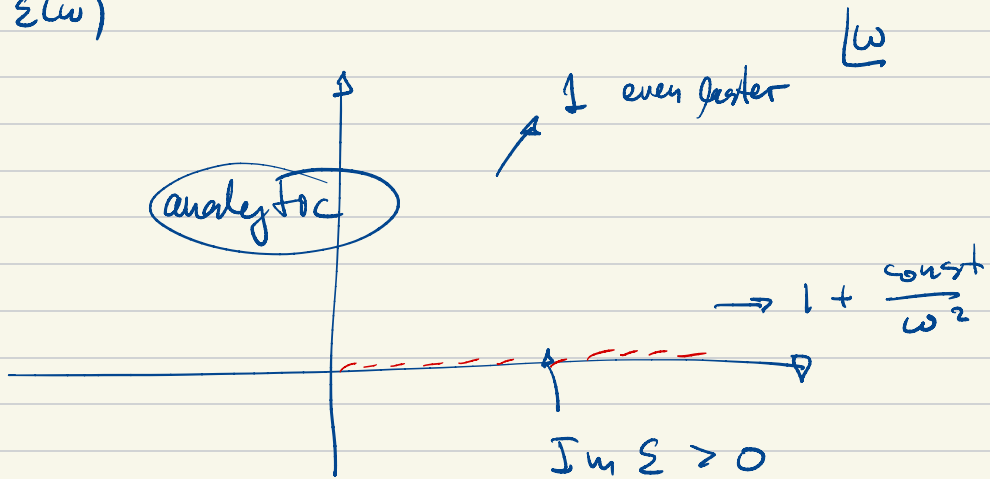
Then

$$m \ddot{r} = e E e^{i\omega t} \Rightarrow$$


$$\Rightarrow r \sim \frac{1}{\omega^2}$$

$$\vec{p} = \sum e \vec{v} \approx \frac{1}{\omega^2} \Rightarrow \vec{D} - \epsilon_0 \vec{E} \sim \frac{1}{\omega^2}$$

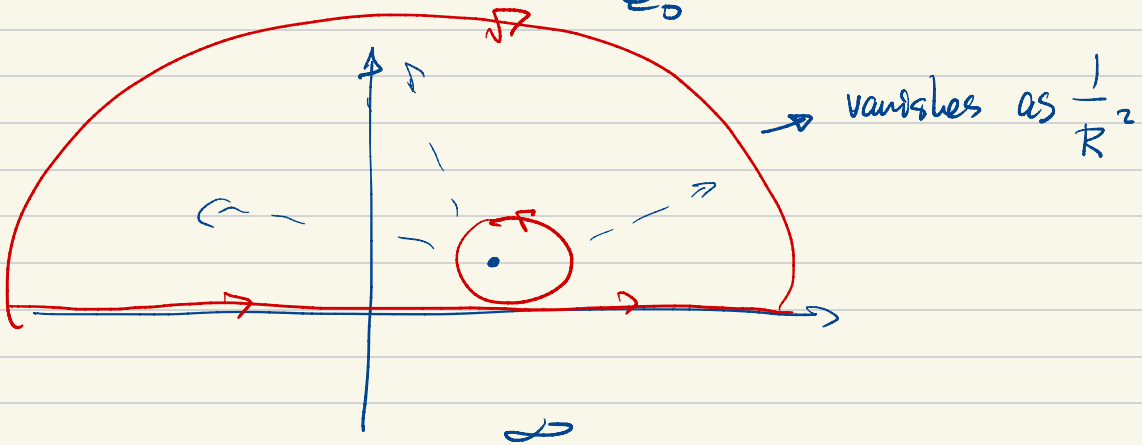
$\epsilon(\omega)$



We will now use Cauchy's theorem to relate $\epsilon'(w)$ and $\epsilon''(w)$ for real w (physical)

$$F(z) = \frac{1}{2\pi i} \oint_{\text{CC}} \frac{F(w')}{w' - z} dw'$$


take $F(z) = \frac{\epsilon(z)}{\epsilon_0} - 1$



$$\frac{\epsilon(z)}{\epsilon_0} - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\epsilon(x)/\epsilon_0 - 1}{x - z} dx$$

Mathematical intrlude

Next we are going to use some properties of regulated singular integrals (or distributions).

Define the "principal value" integral:

$$P \int_{-\infty}^{\infty} \frac{1}{x} f(x) = \lim_{\delta \rightarrow 0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \frac{1}{x} f(x) dx$$

The limit exists for any smooth function $f(x)$.

Define also distributions $\frac{1}{x+i\delta}$ and $\frac{1}{x-i\delta}$

so that

$$I_{+,-}[f(x)] = \lim_{\delta \rightarrow 0} \int \frac{1}{x \pm i\delta} f(x)$$

Again it can be shown that the limit exists.

One more, familiar to us distribution is Dirac δ -function:

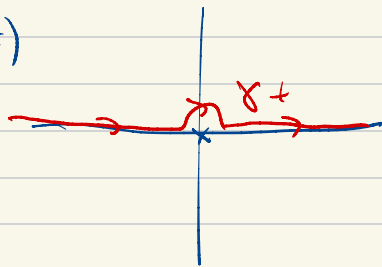
$$\int dx \delta(x) f(x) = f(0). \quad \text{ended here}$$

The distributions are related to each other. Let us find these relations.

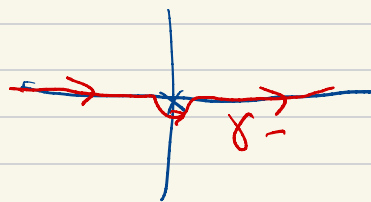
$\frac{1}{x \pm i0}$ distributions can be equivalently

thought of as small contour deformations:

$$\int_{\mathcal{C}} dx \frac{1}{x+i0} f(x) = \int_{\delta^+} dx \frac{1}{x} f(x)$$

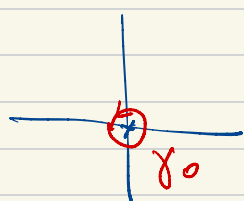


$$\int_{\mathcal{C}} dx \frac{1}{x-i0} f(x) = \int_{\delta^-} dx \frac{1}{x} f(x)$$



Let us consider

$$I_-(f(x)) - I_+(f(x)) = \int_{\gamma_0} \frac{dx}{x} f(x) =$$



$$= 2\pi i f(0) = 2\pi i \int_{\gamma} \delta(x) f(x)$$

Here, even if $f(x)$ is not an analytic function, all what we use is that $f(x)$ is regular at zero.

Since the above is true for any $f(x)$ we conclude that

$$\frac{1}{x-i\delta} - \frac{1}{x+i\delta} = 2\pi i \delta(x) \quad (*)$$

Now consider

$$\int_{\gamma} \left(\frac{1}{x+i\delta} + \frac{1}{x-i\delta} \right) f(x) dx =$$

$$= \left(\int_{-\infty}^{-\Lambda} + \int_{\Lambda}^{\infty} + \int_{-\Lambda}^{\Lambda} \right) \left(\frac{1}{x+i\delta} + \frac{1}{x-i\delta} \right) \rho(x) dx$$

choose $1 \gg \Lambda \gg \delta$, then

$$\begin{aligned} & \left(\int_{-\infty}^{-\Lambda} + \int_{\Lambda}^{\infty} \right) \left(\frac{1}{x+i\delta} + \frac{1}{x-i\delta} \right) \rho(x) dx \approx \\ & \approx \left(\int_{-\infty}^{\Lambda} + \int_{\Lambda}^{\infty} \right) \frac{2}{x} \rho(x) dx = 2P \int_{-\infty}^{\infty} \frac{dx}{x} \rho(x) \end{aligned}$$

At the same time, $\int_{-\Lambda}^{\Lambda} \dots \approx 0$

$$\begin{aligned} & \int_{-\Lambda}^{\Lambda} \left(\frac{1}{x+i\delta} + \frac{1}{x-i\delta} \right) \rho(x) dx = \int_{-\Lambda}^{\Lambda} dx \frac{2x}{x^2+\delta^2} f(0) + \\ & + \text{const.} \int_{-\Lambda}^{\Lambda} dx \frac{2x^2}{x^2+\delta^2} \cdot g'(0) \leq \frac{\text{const}}{\delta^2} \cdot \Lambda^3 \end{aligned}$$

we can choose ϵ and δ so that this expression is arbitrarily small. So we conclude that

$$\frac{1}{x+i\delta} + \frac{1}{x-i\delta} = 2P \frac{1}{x} \quad (*)$$

Combining $*$ and $**$ we get

$$\frac{1}{x-i\delta} = P \frac{1}{x} + \pi i \delta(x).$$

End of mathematical introduction.

We continue with the expression

$$\frac{\epsilon(z)}{\epsilon_0} - 1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\epsilon(x)/\epsilon_0 - 1}{x-z} dx$$

now we take

$$z = \omega + i\delta$$

↑ real ↑ infinitesimal

$$\frac{1}{x - \omega - i\delta} = P \left(\frac{1}{x - \omega} \right) + \pi i \delta(x - \omega)$$

$$\frac{1}{2} \left(\frac{\epsilon(\omega)}{\epsilon_0} - 1 \right) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{\epsilon(x)/\epsilon_0 - 1}{x - \omega} dx$$

Let's take real and imaginary parts:

$$\frac{\epsilon'(\omega)}{\epsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\epsilon''(x)/\epsilon_0}{x - \omega} dx$$

$$\frac{\epsilon''(\omega)}{\epsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\epsilon'(x)/\epsilon_0 - 1}{x - \omega} dx$$

These are the Kramers-Kronig relations

Using symmetry properties we get

$$\frac{\epsilon'(\omega)}{\epsilon_0} = 1 + \frac{2}{\pi} P \int_0^{\infty} x \frac{\epsilon''(x)}{x^2 - \omega^2} dx$$

$$\frac{\epsilon''(\omega)}{\epsilon_0} = -\frac{2\omega}{\pi} P \int_0^{\infty} \frac{\epsilon'(x)/\epsilon_0 - 1}{x^2 - \omega^2} dx$$

Importance of these relations is in the fact that measuring absorption $\epsilon''(\omega)$ allows to reconstruct the entire dispersion of $\epsilon(\omega)$.

Let us consider an example

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{\gamma}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

(resonant absorption on a line)

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{\text{const}}{\omega^2} \quad \text{at } \omega \rightarrow \infty \quad \checkmark$$

$$\frac{\epsilon'}{\epsilon_0} = 1 + \gamma \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \approx$$

$$\approx 1 + \gamma \frac{1}{\omega_0^2 - \omega^2}, \quad \text{when } \gamma \text{ is small,}$$

and $\omega \neq \omega_0$

$$\frac{\epsilon''}{\epsilon_0} = \frac{\gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

Let's check KK: (for $\omega \neq \omega_0$ and γ -small)

$$\int_0^{\infty} \frac{dx}{x^2 - \omega^2} \frac{\gamma x^2}{(\omega_0^2 - x^2)^2 + \gamma^2 x^2} \approx$$

$$\approx \int_{\omega_0 - \gamma}^{\omega_0 + \gamma} \frac{1}{\omega_0^2 - \omega^2} \frac{\gamma x^2}{(\omega_0^2 - x^2)^2 + \gamma^2 x^2}$$

$$\int dz \frac{\gamma \omega_0^2}{(\epsilon \omega_0)^2 z^2 + \gamma^2 \omega_0^2} \approx \lambda \Rightarrow$$

$$\Rightarrow \epsilon' = \epsilon_0 \left(\frac{1}{\omega_0^2 - \omega^2} \right)$$

• Let's come back to plane waves. $\vec{E} = E \cdot e^{i(\vec{k}\vec{r} - \omega t)}$

Since we derived all equations in frequency space, the relations between k and ω still hold:

$$i\omega \mu(\omega) \vec{H} = c \vec{\nabla} \times \vec{E}$$

$$i\omega \epsilon(\omega) \vec{E} = -c \vec{\nabla} \times \vec{H} \Rightarrow$$

$$\vec{k}^2 = \epsilon(\omega) \mu \frac{\omega^2}{c^2}$$

Let us first assume that ϵ is real. Then group velocity of

light is given by $u = \frac{d\omega}{dk} = \frac{c}{d(n\omega)/d\omega}$

where $n(\omega) = \sqrt{\epsilon(\omega)\mu}$.

If ϵ is imaginary k develops imaginary part. Assume $\vec{k} \sim (k_x, 0, 0)$

$$k_x = \sqrt{\epsilon\mu} \frac{\omega}{c} \equiv [n(\omega) + i\alpha(\omega)] \frac{\omega}{c}$$

↑
extinction
coefficient.

KK-like relations can be also derived for n and α .

The key insight is to show that $\sqrt{\epsilon(\omega)\mu}$ has the same analytic properties as $\epsilon(\omega)$ and then use the same logic. This will be done in homework.

• For now assume that $\frac{\epsilon}{\epsilon_0} - 1 \ll 1$,

then $\sqrt{\epsilon \mu}$ has no zeros, and we can write

$$\frac{\hat{n}}{n_0} = \frac{\sqrt{\epsilon \mu}}{\sqrt{\epsilon_0 \mu_0}} = 1 + \frac{\Delta \epsilon}{2} = 1 + \underbrace{\frac{\Delta \epsilon'}{\epsilon^2}}_{n(\omega)} + i \underbrace{\frac{\Delta \epsilon''}{2\epsilon_0}}_{\kappa(\omega)}$$

$$\frac{\epsilon'(\omega)}{\epsilon_0} = 1 + \frac{2}{\pi} P \int_0^{\infty} \frac{x \epsilon''(x)}{x^2 - \omega^2} dx \Rightarrow$$

$$n(\omega) = 1 + \frac{2}{\pi} P \int_0^{\infty} \frac{x \kappa(x)}{x^2 - \omega^2} dx$$

By measuring absorption and applying the formula we get refraction for any $n(\omega)$!

• Radiation in media

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Common types of radiation:

- Bremsstrahlung
- Synchrotron
- Cyclotron
- Cherenkov
- transitions

Bremsstrahlung - "braking" radiation - due to charged particles decelerated in the medium

$$\frac{d\varepsilon}{dt} = \frac{q^2}{6\pi\epsilon_0 c^3} \gamma^6 \left[|\dot{\vec{\beta}}|^2 - |\vec{\beta} \times \dot{\vec{\beta}}|^2 \right]$$

Synchrotron radiation: $\vec{\beta} \perp \dot{\vec{\beta}}$

(Magnetic fields) $\frac{d\varepsilon}{dt} = \frac{q^2}{6\pi\epsilon_0 c^3} \gamma^4 |\dot{\vec{\beta}}|^2$

cyclotron radiation \sim non-relativistic limit
of synchrotron $\gamma \rightarrow 1$

- Bremsstrahlung will be present even in the absence of external electric field because a charged particle moving through a medium will polarize it and decelerate.

- Let's consider this process of deceleration in some detail. We will not consider Bremsstrahlung in details, but we will derive a different type of radiation from it! For now, we will focus on an energy loss of a charged particle.

We start with a non-relativistic case

Let's consider one of the Maxwell equations for the potential:

$$\vec{\nabla} \cdot \vec{D} = \rho \quad \vec{E} = -\vec{\nabla} \phi$$

$$(\epsilon_0 \Delta \phi) = -q \delta(\vec{r} - \vec{v}t)$$

↑
velocity of a charge

Here ϵ_0 is a convolution in time, same as we had above for dispersion relations.

Let's do coordinate Fourier transform:

$$\phi = \int_{-\infty}^{\infty} \phi_k e^{i\vec{k} \cdot \vec{r}} \frac{d^3 k}{(2\pi)^{3/2}}$$

$$\epsilon_0 \phi_k = \frac{q}{k^2} e^{-i\vec{v} \cdot \vec{k}t} \rightarrow \text{single frequency}$$

$\omega \equiv \vec{v} \cdot \vec{k}$

$$\varphi_k = \frac{q}{k^2 \Sigma(\vec{k} \cdot \vec{v})} e^{-i(t + \vec{v} \cdot \vec{k})}, \quad \frac{1}{(2\pi)^{3/2}}$$

$$\vec{E}_k e^{i\vec{k} \cdot \vec{r}} = -i\vec{k} \varphi_k e^{i\vec{k} \cdot \vec{r}}$$

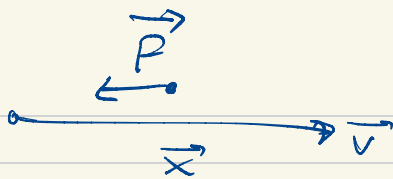
$$\vec{E}(\vec{r}) = \int \vec{E}_k e^{i\vec{k} \cdot \vec{r}} \frac{d^3 k}{(2\pi)^{3/2}}$$

breaking Lorentz \vec{P} is given by
the Lorentz force: $\vec{P} = q \cdot \vec{E}$

$e^{i\vec{v} \cdot \vec{k}}$ cancels $e^{i\vec{k} \cdot \vec{r}}$ at the

position of the particle, and we get

$$\vec{P} = -iq^2 \int d^3 k \frac{\vec{k}}{k^2 \Sigma(\vec{k} \cdot \vec{v}) (2\pi)^3}$$



$$k_x v = \omega \quad Q = \sqrt{k_y^2 + k_z^2}, \quad k^2 = \frac{\omega^2}{v^2} + Q^2$$

$$d^3 k = \frac{d\omega}{v} \cdot 2\pi d$$

$$F = \frac{i q^2}{4\pi^2} \int_0^\infty d\omega \int_0^{Q_0} \frac{Q \omega dQ}{\epsilon(\omega) (Q^2 v^2 + \omega^2)}$$

Q integral is divergent, Q_0 is a short-distance cutoff $F \sim \ln Q_0$.

Integral $d\omega$ has a divergent piece, because $\epsilon(\omega) \rightarrow \epsilon_0$, $\omega \rightarrow \infty$

This piece is not important \rightarrow no

force in the vacuum. Can be subtracted using Principle value prescription.

Only imaginary (odd in ω) part of $\frac{1}{\epsilon}$ contributes.

- Now let's consider the relativistic case, here we need both ϕ and \vec{A} :

$$\Delta A - \frac{\epsilon_0}{c^2} \frac{\partial^2 A}{\partial t^2} = - \frac{qv}{c} \delta(\vec{r} - \vec{v}t)$$

$$\epsilon_0 \left(\Delta \phi - \frac{\epsilon_0}{c^2} \frac{\partial^2 \phi}{\partial t^2} \right) = q \delta(\vec{r} - \vec{v}t)$$

Again Fourier transform $\int \frac{d^3 \Gamma}{(2\pi)^{3/2}} e^{-i\vec{k} \cdot \vec{r}}$

$$A_k = \frac{q}{c} \frac{\vec{v}}{k^2 - \omega^2 \epsilon(\omega)} e^{-i\omega t}$$

$$\phi_k = \frac{q}{\epsilon(\omega)} \frac{1}{k^2 - \omega^2 \epsilon(\omega)} e^{-i\omega t}$$

For electric field we get

$$\vec{E}_k = \frac{i\omega}{c} \vec{A}_k - i\vec{k} \phi_k$$

$$k_x r = \omega \quad Q = \sqrt{k_y^2 + k_z^2} \quad k^2 = \frac{\omega^2}{v^2} + Q^2$$

$$F = qE = \frac{ie^2}{(2\pi)^3} \int d^3k \frac{\frac{\omega v_x}{c^2} - \frac{k_x}{\epsilon(\omega)}}{k^2 - \omega^2 \epsilon(\omega)} =$$

$$= \frac{iq^2}{4\pi^2} \int_{-\infty}^{\infty} d\omega \int_0^{Q_0} \frac{\left(\frac{1}{v^2} - \frac{\epsilon(\omega)}{c^2} \right) \omega q dq d\omega}{\epsilon \left[Q^2 + \omega^2 \left(\frac{1}{v^2} - \frac{\epsilon(\omega)}{c^2} \right) \right]}$$

When $c \rightarrow \infty$ we get non-relativistic result ✓